JGP-Vol.2, n. 1, 1985

Closed geodesics on Riemannian manifolds via the heat flow

S.K. OTTARSSON

Mathematics Institute
University of Warwick
Coventry CV4 7 AL

Abstract. *The main topic discussed in this paper is the following question: Given a Riemannian manifold M and a closed C^1 curve $f : S^1 \to M$ does there exist a (unique) solution of the heat equation $\partial_t f_t = \tau(f_t)$ defined for all $t \geqslant 0$ which is continuous at $t = 0$ along with its first S^1-derivative and which coincides with f at $t = 0$.*

1. INTRODUCTION

In [1] J. Eells and J.H. Sampson treated the more general question where the domain of f is any compact Riemannian manifold (rather than the special case studied here where the domain is the unit circle S^1). They gave an affirmative answer in the case where M has non-positive Riemannian curvature and satisfies some further conditions which were given in terms of an embedding of M in some Euclidean space. These further conditions are always satisfied when M is compact. Then they proved that if M is non-compact but satisfying a further condition (again in terms of the embedding) every such solution will be bounded (i.e. will have its image contained in a compact subset of M), and finally that if M has non-positive Riemannian curvature and f_t is a bounded solution of the heat equation then there is a sequence $t_1, t_2, \ldots$ of t-values such that the mappings

Key-Words: Harmonic Maps.
1980 Mathematics Subject Classification: *53 C 22, 58 G 11.*

f_{t_k} converge uniformly along with their first order space derivatives to a harmonic mapping.

Here the case of closed curves will be approached from a slightly different angle. First for a given closed C^1 curve f a condition on M will be given which will ensure that any solution of the heat equation, which is continuous at $t = 0$ along with its first S^1-derivative and which coincides with f at $t = 0$ has its image contained in a fixed compact set. This condition is different from that in [1] and does not depend on an embedding. Then assuming that that is the case (i.e. all solutions uniformly bounded) the existence of a unique solution defined for all $t \geqslant 0$ will be proved. The proof of this will follow closely the proof of the corresponding result in [1] but the conditions on M (in particular the curvature restriction) used there will not be necessary. Finally it will be proved that this solution subconverges to a closed geodesic, again following [1] but without the curvature restriction.

The assertion that the results of [1] hold for closed curves on compact Riemannian manifolds without curvature restrictions was made in «Variational theory in fibre bundles» by J. Eells and J.H. Sampson, Proc. US-Japan Sem. Diff. Geo. Kyoto (1965) 22 - 23 and (with a proof different from the one given here) in «On harmonic mappings» by J.H. Sampson, Istituto Nazionale di Alta Matematica Francesco Severi Symposia Matematica Vol. XXVI (1982).

The material is arranged in sections as follows: The notation used is fixed in Section 2 which also contains some basic definitions and results in differential geometry. The definitions of energy, tension field etc. are given in Section 3, along with some fundamental properties of solutions of the heat equation. The condition for boundedness of solutions is given in Section 4 and Section 5 has some results about when, in terms of the geometry of the manifold, such a condition might be fulfilled. The proof of existence for all $t \geqslant 0$ is in Section 6 and the proof of subconvergence of the solution to a closed geodesic is in Section 7.

This paper has benefitted from valuable advice and guidance from Professor J. Eells.

2. NOTATION

Throughout M will denote a complete Riemannian manifold. $\langle u, v\rangle$ is the inner product of two tangent vectors u, v at the same point on M and $|v| = \langle v, v\rangle^{\frac{1}{2}}$ the length of v. d is the distance function on M and $B_r(x)$ the open ball centred at x with radius r.

The unit circle S^1 will always be parametrized by the central angle θ. If f is a mapping with domain $S^1 \times I$ where I is a subset of $\mathbb{R}$, for a fixed t in I f_t is the mapping with domain S^1 given by $f_t(\theta) = f(\theta, t)$. $f : S^1 \to \mathbb{R}$ will sometimes be identified with $f \circ (\theta \to e^{i\theta}) : \mathbb{R} \to \mathbb{R}$. ∂_θ denotes differentiation with respect to θ, $\partial_\theta f = f_* \partial_\theta$ where f_* is the differential of f, similarly for ∂_t.

The symbol ∇ will be used for the Levi-Civita connection on M and for the induced connection on the vector fields along a smooth mapping f into M. The following facts about the connection (from [2]) will be needed.

Let X, Y, Z be vector fields on M. The torsion tensor T defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \tag{2.1}$$

satisfies $T \equiv 0$. The curvature tensor R is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

Let N be a smooth manifold, $f : N \to M$ smooth, A, B vector fields on N and X, Y vector fields along f. Then

$$0 = T(f_*A, f_*B) = \nabla_A f_*B - \nabla_B f_*A - f_*[A, B] \tag{2.2}$$

$$R(f_*A, f_*B)X = \nabla_A \nabla_B X - \nabla_B \nabla_A X - \nabla_{[A, B]} X \tag{2.3}$$

$$A\langle X, Y\rangle = \langle \nabla_A X, Y\rangle + \langle X, \nabla_A Y\rangle. \tag{2.4}$$

3. THE HEAT EQUATION

DEFINITIONS. For a C^1 curve $f : S^1 \to M$ its energy density $e(f)$ is the function $S^1 \to \mathbb{R}$ defined by $e(f)(\theta) = \frac{1}{2} | \partial_\theta f(\theta)|^2$. The energy $E(f)$ of f is defined by $E(f) = \int_0^{2\pi} e(f)(\theta)\, d\theta$. The length $L(g)$ of a C^1 curve g is the integral of the length of its tangent vector over its domain. When f is C^2 its tension field $\tau(f)$ is the vector field along f given by $\tau(f) = \nabla_{\partial_\theta} \partial_\theta f$. f is a geodesic iff its tension field vanishes.

LEMMA 3A. *Let $f_t : S^1 \to M$ be a smooth family of closed curves for t in some open interval. Put $E(t) = E(f_t)$. Then*

$$\partial_t E(t) = -\int_0^{2\pi} \langle \tau(f_t)(\theta), \partial_t f_t(\theta)\rangle\, d\theta.$$

Proof. By definition

$$E(t) = \int_0^{2\pi} \frac{1}{2} \langle \partial_\theta f_t(\theta), \partial_\theta f_t(\theta) \rangle \, \mathrm{d}\theta$$

so by (2.4) and (2.2)

$$\partial_t E(t) = \int_0^{2\pi} \langle \nabla_{\partial_t} \partial_\theta f_t(\theta), \partial_\theta f_t(\theta) \rangle \, \mathrm{d}\theta =$$

$$= \int_0^{2\pi} \langle \nabla_{\partial_\theta} \partial_t f_t(\theta), \partial_\theta f_t(\theta) \rangle \, \mathrm{d}\theta.$$

Further

$$0 = \int_0^{2\pi} \partial_\theta \langle \partial_t f_t(\theta), \partial_\theta f_t(\theta) \rangle \, \mathrm{d}\theta =$$

$$= \int_0^{2\pi} [\langle \nabla_{\partial_\theta} \partial_t f_t(\theta), \partial_\theta f_t(\theta) \rangle +$$

$$+ \langle \partial_t f_t(\theta), \nabla_{\partial_\theta} \partial_\theta f_t(\theta) \rangle] \, \mathrm{d}\theta,$$

and from this

$$\int_0^{2\pi} \langle \nabla_{\partial_\theta} \partial_t f_t(\theta), \partial_\theta f_t(\theta) \rangle \, \mathrm{d}\theta =$$

$$= -\int_0^{2\pi} \langle \nabla_{\partial_\theta} \partial_\theta f_t(\theta), \partial_t f_t(\theta) \rangle \, \mathrm{d}\theta.$$

COROLLARY. *If f_t is a smooth solution of the heat equation*

(3.1) $$\partial_t f_t = \tau(f_t)$$

then

(3.2) $$\partial_t E(t) = -\int_0^{2\pi} | \partial_t f_t |^2 \, \mathrm{d}\theta.$$

Therefore $\partial_t E(t) \leqslant 0$ *with* $\partial_t E(t) = 0$ *only when* f_t *is a geodesic.*

The following propositions are special cases of Propositions 2(B) and 6(B) of [1].

PROPOSITION. *Every* C^2 *curve* $f: S^1 \to M$ *which satisfies* $\tau(f) = 0$ *is smooth.*

PROPOSITION. *If* $(\theta, t) \to f_t(\theta)$ *is a map of* $S^1 \times (t_0, t_1) \to M$ *which is* C^1 *on the product manifold and* C^2 *on* S^1 *for each* t, *and if that map satisfies* (3.1), *then it is smooth.*

4. BOUNDEDNESS CONDITIONS

Solutions of the heat equation that are not bounded exist as is shown by the following example from [1].

Let N be the manifold obtained by revolving the graph of $v(u) = 1 + e^{-u}$ around the u-axis. If f satisfies $\partial_\theta u = 0$, $\phi = \theta$ (ϕ revolution angle) then so does the solution f_t for any subsequent time. The heat equation reduces to $\partial_t u = \dfrac{e^u + 1}{e^{2u} + 1}$. Thus $e^u + u - 2 \log (e^u + 1) = t + \text{const}$, in particular $u \to \infty$ as $t \to \infty$. However, if the length of f is less than 2π one would expect the solution to be bounded.

Fix a C^1 curve $f: S^1 \to M$. Throughout this section $f_t : S^1 \to M$ is a solution of the heat equation for $0 \leqslant t < b \leqslant \infty$ which is continuous along with $\partial_\theta f_t$ at $t = 0$ and which coincides with f at $t = 0$. As before $E(t) = E(f_t)$.

DEFINITION. Let $c > 0$ and U be an open set in M. U has the property $P(c)$ if for every C^2 curve $g : S^1 \to U$

$$(4.1) \qquad cE(g) \leqslant \int_0^{2\pi} |\tau(g)|^2 d\theta.$$

Put $m(f, c) = (2\pi)^{-\frac{1}{2}} E(f) + (3\pi^{\frac{1}{2}} + 2(2\pi)^{-\frac{1}{2}}) E(f)^{\frac{1}{2}} + 4(2\pi)^{-\frac{1}{2}} c^{-1} E(f)^{\frac{1}{4}}$.

THEOREM 4A. *Let* M *satisfy the following condition: There is a compact* $K \subset M$ *such that for every* x *in the complement of* K *there exist real numbers* k, c *with* $c > 0$ *and* $k > m(f, c)$ *such that* $B_k(x)$ *has the property* $P(c)$. *Then the image of* f_t *is bounded, and further if* $b = \infty$ *and if the image of some* f_{t_0} *lies in* $M \setminus K$ *then there exists a* $p \in M$ *such that* $\lim_{t \to \infty} f_t \equiv p$.

Proof. For every closed C^1 curve $g : S^1 \to M$ one has the inequalities

$$\operatorname{diam} g(S^1) \leqslant \frac{1}{2} L(g) \leqslant \pi^{\frac{1}{2}} E(g)^{\frac{1}{2}}. \tag{4.2}$$

By the corollary to Lemma 3A $E(t)$ is a non-increasing function and is therefore bounded by $E(f)$. Bearing that and (4.2) in mind it is seen that if all the curves f_t with $t \in (0, b)$ intersect K then the image of the solution is bounded.

So suppose that the image of f_{t_0} lies in $M \setminus K$ for some $t_0 \in (0, b)$. Fix $\theta_0 \in S^1$. By hypothesis there exist $c > 0$ and $k > m(f, c)$ such that $B_k(f_{t_0}(\theta_0))$ has the property $P(c)$. It is easily seen that $f_{t_0}(S^1) \subset B_k(f_{t_0}(\theta_0))$. Put $t_1 = \sup\{t' > t_0 : f_t(S^1) \subset B_k(f_{t_0}(\theta_0))$ for all $t \in [t_0, t')\}$ and suppose for a contradiction that $t_1 < b$. It is easy to show that

$$\sup_{\theta \in S^1} d(f_{t_1}(\theta), f_{t_0}(\theta)) \leqslant \pi^{\frac{1}{2}} [E(t_0)^{\frac{1}{2}} + E(t_1)^{\frac{1}{2}}] + \tag{4.3}$$

$$+ \frac{1}{2\pi} \int_0^{2\pi} \int_{t_0}^{t_1} |\partial_t f_t(\theta)| \, dt \, d\theta.$$

Estimate of the last term in (4.3): By (3.2)

$$\frac{1}{2\pi} \int_0^{2\pi} \int_{t_0}^{t_1} |\partial_t f_t(\theta)| \, dt \, d\theta \leqslant$$

$$\leqslant (2\pi)^{-\frac{1}{2}} \int_{t_0}^{t_1} \left(\int_0^{2\pi} |\partial_t f_t(\theta)|^2 d\theta \right)^{\frac{1}{2}} dt = \tag{4.4}$$

$$= (2\pi)^{-\frac{1}{2}} \int_{t_0}^{t_1} (-\partial_t E(t))^{\frac{1}{2}} \, dt.$$

Put $T_1 = \{t \geqslant t_0 : -\partial_t E(t) \geqslant 1\}$, $T_2 = \{t \geqslant t_0 : E(t)^{\frac{1}{2}} \leqslant -\partial_t E(t) \leqslant 1\}$, $T_3 = \{t \geqslant t_0 : -\partial_t E(t) \leqslant E(t)^{\frac{1}{2}}\}$. From (4.4)

$$\frac{1}{2\pi} \int_0^{2\pi} \int_{t_0}^{t_1} |\partial_t f_t(\theta)| \, dt \, d\theta \leqslant \tag{4.5}$$

$$\leqslant (2\pi)^{-\frac{1}{2}} \int_{[t_0,t_1]\cap T_1} (-\partial_t E(t))^{\frac{1}{2}}\,\mathrm{d}t +$$

(4.5)
$$+ (2\pi)^{-\frac{1}{2}} \int_{[t_0,t_1]\cap T_2} (-\partial_t E(t))^{\frac{1}{2}}\,\mathrm{d}t +$$

$$+ (2\pi)^{-\frac{1}{2}} \int_{[t_0,t_1]\cap T_3} (-\partial_t E(t))^{\frac{1}{2}}\,\mathrm{d}t.$$

I
$$(2\pi)^{-\frac{1}{2}} \int_{[t_0,t_1]\cap T_1} (-\partial_t E(t))^{\frac{1}{2}}\,\mathrm{d}t \leqslant$$

(4.6)
$$\leqslant (2\pi)^{-\frac{1}{2}} \int_{[t_0,t_1]\cap T_1} (-\partial_t E(t))\,\mathrm{d}t \leqslant$$
$$\leqslant (2\pi)^{-\frac{1}{2}} (E(t_0) - E(t_1)).$$

II Suppose $[t', t''] \subset T_2$ i.e. $E(t)^{\frac{1}{2}} \leqslant -\partial_t E(t) \leqslant 1$ for all $t \in [t', t'']$. Then

$$1 \leqslant -\frac{\partial_t E(t)}{E(t)^{\frac{1}{2}}} = -2\,\partial_t (E^{\frac{1}{2}}) \quad \text{so}$$

$$t'' - t' \leqslant -2\int_{t'}^{t''} \partial_t (E^{\frac{1}{2}})\mathrm{d}t = 2(E(t')^{\frac{1}{2}} - E(t'')^{\frac{1}{2}}).$$

Therefore, the sum of the lengths of the intervals making up $[t_0, t_1] \cap T_2$ is no greater than $2(E(t_0)^{\frac{1}{2}} - E(t_1)^{\frac{1}{2}})$ and so

(4.7)
$$(2\pi)^{-\frac{1}{2}} \int_{[t_0,t_1]\cap T_2} (-\partial_t E(t))^{\frac{1}{2}}\,\mathrm{d}t \leqslant 2(2\pi)^{-\frac{1}{2}} (E(t_0)^{\frac{1}{2}} - E(t_1)^{\frac{1}{2}}).$$

III Since $B_k(f_{t_0}(\theta_0))$ has the property $P(c)$ one has

$$cE(t) \leqslant \int_0^{2\pi} |\tau(f_t)(\theta)|^2 \mathrm{d}\theta = \int_0^{2\pi} |\partial_t f_t(\theta)|^2 \mathrm{d}\theta = -\partial_t E(t)$$

and therefore

$$\frac{\partial_t E(t)}{E(t)} \leqslant -c, \quad \text{for all} \quad t \in [t_0, t_1).$$

From this it follows that for $t \in [t_0, t_1)$

$$E(t) \leqslant E(t_0) e^{-c(t-t_0)}. \tag{4.8}$$

Therefore

$$(2\pi)^{-\frac{1}{2}} \int_{[t_0,t_1] \cap T_3} (-\partial_t E(t))^{\frac{1}{2}} \mathrm{d}t \leqslant (2\pi)^{-\frac{1}{2}} \int_{[t_0,t_1] \cap T_3} E(t)^{\frac{1}{4}} \mathrm{d}t \leqslant$$

$$\leqslant (2\pi)^{-\frac{1}{2}} \int_{t_0}^{t_1} E(t_0)^{\frac{1}{4}} \exp\left[\frac{-c(t-t_0)}{4}\right] \mathrm{d}t = \tag{4.9}$$

$$= 4(2\pi)^{-\frac{1}{2}} c^{-1} E(t_0)^{\frac{1}{4}} \left(1 - \exp\left[\frac{-c(t_1-t_0)}{4}\right]\right).$$

Combining (4.3), (4.5), (4.6), (4.7), (4.9)

$$\sup_{\theta \in S^1} \mathrm{d}(f_{t_1}(\theta), f_{t_0}(\theta)) \leqslant \pi^{\frac{1}{2}} [E(t_0)^{\frac{1}{2}} + E(t_1)^{\frac{1}{2}}] + (2\pi)^{-\frac{1}{2}} [E(t_0) -$$

$$- E(t_1)] + 2(2\pi)^{-\frac{1}{2}} [E(t_0)^{\frac{1}{2}} - E(t_1)^{\frac{1}{2}}] + \tag{4.10}$$

$$+ 4(2\pi)^{-\frac{1}{2}} c^{-1} E(t_0)^{\frac{1}{4}} \left(1 - \exp\left[\frac{-c(t_1-t_0)}{4}\right]\right).$$

And further

$$\sup_{\theta \in S^1} \mathrm{d}(f_{t_1}(\theta), f_{t_0}(\theta)) \leqslant (2\pi)^{-\frac{1}{2}} E(t_0) +$$

$$+ (2\pi^{\frac{1}{2}} + 2(2\pi)^{-\frac{1}{2}}) E(t_0)^{\frac{1}{2}} + 4(2\pi)^{-\frac{1}{2}} c^{-1} E(t_0)^{\frac{1}{4}}.$$

From this it follows that $f_{t_1}(S^1) \subset B_k(f_{t_0}(\theta_0))$, but that would mean that there is an $\epsilon > 0$ such that $f_{t_1+\tau}(S^1) \subset B_k(f_{t_0}(\theta_0))$ for all $\tau \in [0, \epsilon)$ contradicting the choice of t_1.

For the proof of the last assertion of the theorem observe that by the above proof $f_t(S^1) \subset B_k(f_{t_0}(\theta_0))$ for all $t \in [t_0, \infty)$ and therefore the inequality (4.8) will hold for all $t \in [t_0, \infty)$. Further, for all $t', t'' \in [t_0, \infty)$ the inequality (4.10) will hold for t', t'' in place of t_0, t_1. For a fixed θ (4.8) and (4.10) show that $f_t(\theta)$ converges to a point p as $t \to \infty$. Further, (4.8) shows that $E(t)$ and therefore (by (4.2)) $L(f_t)$ converge to zero, which shows that $\lim_{t \to \infty} f(\theta)$ is independent of θ.

Remark. In the case of $M = \mathbb{R}^n$ it is easy to show that the inequality $E(f) \leqslant \int_0^{2\pi} |\tau(f)|^2 \, d\theta$ holds for every closed C^2 curve $f : S^1 \to \mathbb{R}^n$:

First consider real valued f. For any $f \in L^1(S^1)$ the Fourier coefficients of f are defined by the formula

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, e^{-in\theta} \, d\theta \qquad n \in \mathbb{Z}$$

and one has the Parseval theorem

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)\, \overline{\hat{g}(n)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, \overline{g(\theta)} \, d\theta$$

whenever f, $g \in L^2(S^1)$; the series on the left converges absolutely. When f is C^2 one has

$$\widehat{\partial_\theta f}(n) = in\hat{f}(n), \quad \widehat{\partial_\theta^2 f}(n) = -n^2 \hat{f}(n) \qquad n \in \mathbb{Z}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |\partial_\theta f(\theta)|^2 \, d\theta = \sum_{n=-\infty}^{\infty} |\widehat{\partial_\theta f}(n)|^2 = \sum_{n=-\infty}^{\infty} n^2 |\hat{f}(n)|^2$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |\partial_\theta^2 f(\theta)|^2 \, d\theta = \sum_{n=-\infty}^{\infty} n^4 |\hat{f}(n)|^2.$$

Therefore $\| \partial_\theta f \|_2^2 \leqslant \| \partial_\theta^2 f \|_2^2$ and the same inequality holds for $\mathbb{R}^n$-valued f because it holds for each component.

5. MORE ON BOUNDEDNESS CONDITIONS

In this section a demonstration of the inequality

$$\frac{1}{2\pi^2} E(f) \leqslant \int_0^{2\pi} |\tau(f)|^2 \, d\theta$$

for closed curves satisfying certain assumptions is given following a proof of the Synge-formula given on pp. 122 - 3 in [2].

DEFINITION. If A is a subset of M define

$$\kappa_A = \sup\{\langle R(X, Y)Y, X\rangle : X, Y \in T_pM, |X| = |Y| = 1, p \in A\}.$$

Then for all $p \in A, X, Y \in T_pM, \langle R(X, Y)Y, X\rangle \leqslant \kappa_A |X|^2 |Y|^2$.

THEOREM 5A. *Let $f: S^1 \to M$ be a non-constant closed C^2 curve and assume w.l.o.g. that the maximum of the energy density occurs at $\theta = 0$. Suppose that the image of f is contained in a ball $B_r(f(0))$ satisfying the following: The exponential map $\exp_{f(0)}: T_{f(0)}M \to M$ restricted to $B_r(0)$ in $T_{f(0)}M$ is a diffeomorphism onto $B_r(f(0))$ and if $\kappa_B > 0$ the radius r satisfies $r < (2\kappa_B^{\frac{1}{2}})^{-1}$. Then the inequality $\frac{1}{2\pi^2} E(f) \leqslant \int_0^{2\pi} |\tau(f)|^2 \, d\theta$ holds.*

Remark. The inequality is of course trivial when f is constant.

During the proof consider θ in the interval $(0, \theta_0)$ where θ_0 is the first $\theta > 0$ such that $f(\theta) = f(0)$. Put $B = B_r(f(0))$. To begin with a few definitions.

Define $f^*: S^1 \to T_{f(0)}M$ by $f^* = \exp_{f(0)}^{-1} \circ f$ and a family of geodesics g by $g(\theta, t) = \exp_{f(0)}(t \cdot f^*(\theta))$. For a fixed θ $g^\theta(\cdot) = g(\theta, \cdot)$ is a geodesic from $f(0)$ to $f(\theta)$ $(g^\theta(0) = f(0), g^\theta(1) = f(\theta))$.

Define three vector fields along the map $g: X = \partial_t g^\theta$, $Y = \partial_\theta g^\theta$ and $\widetilde{Y} = Y - \langle Y, X\rangle X / |X|^2$.

Define $L(\theta)$ to be the length of g^θ. $L(\theta) = |\partial_t g^\theta| = |X|$.

Step I

$\lim_{\theta \to 0^+} X/|X|(\theta, t)$ exists and equals the unit vector in the direction of $\partial_\theta f|_{\theta = 0}$.

Proof. Identify $T_{f(0)}M$ and all its tangent spaces with $\mathbb{R}^n$ and so let $J_v(u)$ be the vector u translated to $(T_{f(0)}M)_v$. One has

$$\frac{X}{|X|}(\theta, t) = \exp_{f(0)*}\left(J_{tf^*(\theta)} \frac{f^*(\theta)}{|f^*(\theta)|}\right)$$

(compare Gauß-lemma p. 136 [2]). $\lim_{\theta \to 0^+} \frac{f_*^*(\theta)}{|f^*(\theta)|}$ exists (in $T_{f(0)}M$) and equals the unit vector in the direction of $\partial_\theta f|_{\theta=0}$ and so $\lim_{\theta \to 0^+} \frac{X}{|X|}(\theta, t)$ exists and is equal to the same vector.

$$(5.1)\quad \begin{cases} \text{Since } g(\theta, 0) \equiv f(0),\ Y(\theta, 0) \equiv 0 \text{ and also } \nabla_{\partial_\theta} Y(\theta, 0) \equiv 0 \\ \text{Since } g(\theta, 1) = f(\theta),\ Y(\theta, 1) = \partial_\theta f(\theta) \text{ and also } \nabla_{\partial_\theta} Y(\theta, 1) = \tau(f)(\theta). \end{cases}$$

The vector field $\widetilde{Y}$ along g (the component of Y orthogonal to X) is C^1 for $\theta \in (0, \theta_0)$ and $\widetilde{Y}(\theta, 0) \equiv 0$ by (5.1).

Step II

$$(5.2)\qquad \partial_\theta L(\theta) = \left\langle \partial_\theta f(\theta), \frac{X}{|X|}(\theta, 1) \right\rangle .$$

Proof. By (2.2)

$$(5.3)\qquad \nabla_{\partial_\theta} X = \nabla_{\partial_t} Y.$$

Using this and $L(\theta) = \int_0^1 |X|\, dt$ there follows

$$\partial_\theta L(\theta) = \int_0^1 \frac{\partial_\theta \langle X, X\rangle}{2|X|}\, dt = \int_0^1 \frac{\langle \nabla_{\partial_\theta} X, X\rangle}{|X|}\, dt =$$

$$= \int_0^1 \frac{\langle \nabla_{\partial_t} Y, X\rangle}{|X|}\, dt = \int_0^1 \partial_t \left\langle Y, \frac{X}{|X|} \right\rangle dt =$$

$$= \left\langle Y, \frac{X}{|X|} \right\rangle (\theta, t)\big|_{t=0,1} = \left\langle \partial_\theta f(\theta), \frac{X}{|X|}(\theta, 1) \right\rangle \text{ by (5.1).}$$

COROLLARY.

$$(5.4)\qquad \lim_{\theta \to 0^+} \partial_\theta L(\theta) = |\partial_\theta f(0)|.$$

This follows immediately from (5.2) *and Step I.*

Step III

(5.5)
$$\partial_\theta^2 L(\theta) = \int_0^1 \frac{1}{|X|} \left(\langle \nabla_{\partial_t} \widetilde{Y}, \nabla_{\partial_t} \widetilde{Y}\rangle - \langle R(X, \widetilde{Y})\widetilde{Y}, X\rangle\right) \mathrm{d}t + \\ + \left\langle \tau(f)(\theta), \frac{X}{|X|}(\theta, 1)\right\rangle .$$

Proof. By (5.3)

(5.6)
$$\partial_\theta\left(\frac{1}{|X|}\right) = \partial_\theta \langle X, X\rangle^{-\frac{1}{2}} = -\langle X, X\rangle^{-\frac{3}{2}} \langle \nabla_{\partial_\theta} X, X\rangle = \\ = -\frac{1}{|X|^3} \langle \nabla_{\partial_t} Y, X\rangle.$$

From above $\partial_\theta L(\theta) = \int_0^1 \frac{1}{|X|} \langle \nabla_{\partial_t} Y, X\rangle \, \mathrm{d}t$, therefore using (5.6) and (5.3) one obtains

(5.7)
$$\partial_\theta^2 L(\theta) = \int_0^1 \left(\frac{1}{|X|} \left(\partial_\theta \langle \nabla_{\partial_t} Y, X\rangle\right) + \left(\partial_\theta \frac{1}{|X|}\right) \langle \nabla_{\partial_t} Y, X\rangle \right) \mathrm{d}t = \\ = \int_0^1 \left(\frac{1}{|X|} \left(\langle \nabla_{\partial_\theta} \nabla_{\partial_t} Y, X\rangle + \langle \nabla_{\partial_t} Y, \nabla_{\partial_\theta} X\rangle\right) - \frac{1}{|X|^3} \langle \nabla_{\partial_t} Y, X\rangle^2 \right) \mathrm{d}t = \\ = \int_0^1 \left(\frac{1}{|X|} \left(\langle \nabla_{\partial_\theta} \nabla_{\partial_t} Y, X\rangle + \langle \nabla_{\partial_t} Y, \nabla_{\partial_t} Y\rangle\right) - \frac{1}{|X|^3} \langle \nabla_{\partial_t} Y, X\rangle^2 \right) \mathrm{d}t.$$

Since for a fixed θ $g(\theta, \cdot)$ is a geodesic $\nabla_{\partial_t} X = 0$ and $\partial_t |X| = 0$. Therefore

(5.8)
$$\nabla_{\partial_t}\left(\frac{1}{|X|^2} \langle Y, X\rangle X \right) = \frac{1}{|X|^2} \left(\partial_t \langle Y, X\rangle\right) X.$$

Also, since $\langle \widetilde{Y}, X\rangle = 0$

(5.9)
$$\langle \nabla_{\partial_t} \widetilde{Y}, X\rangle = \partial_t \langle \widetilde{Y}, X\rangle = 0$$

Now

$$Y = \widetilde{Y} + \frac{1}{|X|^2} \langle Y, X \rangle X$$

and

$$\nabla_{\partial_t} Y = \nabla_{\partial_t} \widetilde{Y} + \nabla_{\partial_t}\left(\frac{1}{|X|^2} \langle Y, X \rangle X\right) =$$

$$= \nabla_{\partial_t} \widetilde{Y} + \frac{1}{|X|^2} (\partial_t \langle Y, X \rangle) X$$

by (5.8). From this it follows that

$$\langle \nabla_{\partial_t} Y, \nabla_{\partial_t} Y \rangle = \langle \nabla_{\partial_t} \widetilde{Y}, \nabla_{\partial_t} \widetilde{Y} \rangle + \frac{1}{|X|^2} (\partial_t \langle Y, X \rangle) \cdot 2 \langle \nabla_{\partial_t} \widetilde{Y}, X \rangle + $$
$$+ \frac{1}{|X|^4} (\partial_t \langle Y, X \rangle)^2 \langle X, X \rangle.$$

The second term in this expression vanishes by (5.9), the last one equals $\frac{1}{|X|^2} (\langle \nabla_{\partial_t} Y, X \rangle)^2$. In other words

$$\langle \nabla_{\partial_t} Y, \nabla_{\partial_t} Y \rangle = \langle \nabla_{\partial_t} \widetilde{Y}, \nabla_{\partial_t} \widetilde{Y} \rangle) + \frac{1}{|X|^2} (\langle \nabla_{\partial_t} Y, X \rangle)^2.$$

Substituting this in (5.7) gives

$$\partial_\theta^2 L(\theta) = \int_0^1 \frac{1}{|X|} (\langle \nabla_{\partial_\theta} \nabla_{\partial_t} Y, X \rangle + \langle \nabla_{\partial_t} \widetilde{Y}, \nabla_{\partial_t} \widetilde{Y} \rangle)\, dt. \tag{5.10}$$

By (2.3) $\nabla_{\partial_\theta} \nabla_{\partial_t} Y = \nabla_{\partial_t} \nabla_{\partial_\theta} Y - R(X, Y) Y$. Therefore $\langle \nabla_{\partial_\theta} \nabla_{\partial_t} Y, X \rangle =$ $= \langle \nabla_{\partial_t} \nabla_{\partial_\theta} Y, X \rangle - \langle R(X, Y) Y, X \rangle = \partial_t \langle \nabla_{\partial_\theta} Y, X \rangle - \langle R(X, Y) Y, X \rangle$.
Substituting this in (5.10) gives

$$\partial_\theta^2 L(\theta) = \int_0^1 \left(\partial_t \left\langle \nabla_{\partial_\theta} Y, \frac{X}{|X|} \right\rangle + \frac{1}{|X|} (\langle \nabla_{\partial_t} \widetilde{Y}, \nabla_{\partial_t} \widetilde{Y} \rangle + \langle R(X, Y) Y, X \rangle)\right) dt. \tag{5.11}$$

From p. 91 [2] $\langle R(X,Y)Z,U\rangle = -\langle R(X,Y)U,Z\rangle$ and therefore $\langle R(X,Y)Z,Z\rangle = 0$. One has also $R(X,Y)Z = -R(Y,X)Z$ and therefore $R(X,X)Z = 0$.

Put $h(\theta,t) = \dfrac{1}{|X|^2}\langle Y,X\rangle$. Then $\widetilde{Y} = Y - h.X$.

$$\langle R(\widetilde{Y},X)X,\widetilde{Y}\rangle = \langle R(Y,X)X,\widetilde{Y}\rangle - \langle R(h.X,X)X,\widetilde{Y}\rangle =$$

$$= \langle R(Y,X)X,Y\rangle - \langle R(Y,X)X,h.X\rangle = \langle R(Y,X)X,Y\rangle.$$

This shows that $\langle R(X,\widetilde{Y})\widetilde{Y},X\rangle = \langle R(X,Y)Y,X\rangle$. This changes (5.11) into

$$\partial_\theta^2 L(\theta) = \int_0^1 \frac{1}{|X|}(\langle\nabla_{\partial_t}\widetilde{Y},\nabla_{\partial_t}\widetilde{Y}\rangle - \langle R(X,\widetilde{Y})\widetilde{Y},X\rangle)\,\mathrm{d}t +$$

$$+\int_0^1 \partial_t\left\langle\nabla_{\partial_\theta}Y,\frac{X}{|X|}\right\rangle \mathrm{d}t = \int_0^1\frac{1}{|X|}(\langle\nabla_{\partial_t}\widetilde{Y},\nabla_{\partial_t}\widetilde{Y}\rangle -$$

$$-\langle R(X,\widetilde{Y})\widetilde{Y},X\rangle)\mathrm{d}t + \left\langle \tau(f)(\theta),\frac{X}{|X|}(\theta,1)\right\rangle$$

(using (5.1)) which is (5.5).

Step IV

The integral $I = \int_0^1 \dfrac{1}{|X|}(\langle\nabla_{\partial_t}\widetilde{Y},\nabla_{\partial_t}\widetilde{Y}\rangle - \langle R(X,\widetilde{Y})\widetilde{Y},X\rangle)\,\mathrm{d}t$ is non-negative.

Proof. $I = \dfrac{1}{|X|}\int_0^1(\langle\nabla_{\partial_t}\widetilde{Y},\nabla_{\partial_t}\widetilde{Y}\rangle - \langle R(X,\widetilde{Y})\widetilde{Y},X\rangle)\,\mathrm{d}t$ since $|X|$ is independent of t.

Obviously if $\kappa_B \leqslant 0$ the integral is non-negative.

Suppose $\kappa_B > 0$. As remarked above $\widetilde{Y}(\theta,0) \equiv 0$. Consequently

$$|\widetilde{Y}|^2(\theta,t) = \langle\widetilde{Y},\widetilde{Y}\rangle(\theta,t) - \langle\widetilde{Y},\widetilde{Y}\rangle(\theta,0) =$$

$$= \int_0^t \partial_t\langle\widetilde{Y},\widetilde{Y}\rangle\,\mathrm{d}t = \int_0^t 2\langle\nabla_{\partial_t}\widetilde{Y},\widetilde{Y}\rangle\,\mathrm{d}t \leqslant$$

$$\leqslant 2\int_0^t |\nabla_{\partial_t}\widetilde{Y}\|\widetilde{Y}|\,\mathrm{d}t \leqslant 2\left(\int_0^1|\nabla_{\partial_t}\widetilde{Y}|^2\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^1|\widetilde{Y}|^2\mathrm{d}t\right)^{\frac{1}{2}}.$$

Therefore,

$$\int_0^1 |\widetilde{Y}|^2 \,\mathrm{d}t \leqslant 2\left(\int_0^1 |\nabla_{\partial_t}\widetilde{Y}|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \left(\int_0^1 |\widetilde{Y}|^2 \,\mathrm{d}t\right)^{\frac{1}{2}}.$$

Or

$$\int_0^1 |\widetilde{Y}|^2 \,\mathrm{d}t \leqslant 4\int_0^1 |\nabla_{\partial_t}\widetilde{Y}|^2 \,\mathrm{d}t. \tag{5.12}$$

Since the radius r of B satisfies $r < (2\kappa_B^{\frac{1}{2}})^{-1}$ one has $|X| \leqslant (2\kappa_B^{\frac{1}{2}})^{-1}$. Therefore

$$\langle R(X, \widetilde{Y})\widetilde{Y}, X\rangle \leqslant \kappa_B |X|^2 |\widetilde{Y}|^2 \leqslant \frac{1}{4}|\widetilde{Y}|^2.$$

Consequently

$$\int_0^1 \langle R(X, \widetilde{Y})\widetilde{Y}, X\rangle \,\mathrm{d}t \leqslant \frac{1}{4}\int_0^1 |\widetilde{Y}|^2 \,\mathrm{d}t \leqslant \int_0^1 |\nabla_{\partial_t}\widetilde{Y}|^2 \,\mathrm{d}t$$

using (5.12). This gives

$$\int_0^1 (|\nabla_{\partial_t}\widetilde{Y}|^2 - \langle R(X, \widetilde{Y})\widetilde{Y}, X\rangle) \,\mathrm{d}t \geqslant 0.$$

By the corollary of Step II $\lim_{\theta \to 0^+} \partial_\theta L(\theta) = |\partial_\theta f(0)|$. From this one obtains

$$\partial_\theta L(\theta) = |\partial_\theta f(0)| + \int_0^\theta \partial_\theta^2 L(\theta')\,\mathrm{d}\theta'.$$

From Steps III and IV there follows

$$\partial_\theta L(\theta) \geqslant |\partial_\theta f(0)| - \int_0^\theta |\tau(f)(\theta')|\,\mathrm{d}\theta'.$$

Since f is a closed curve there must come a θ_1 where $\partial_\theta L(\theta_1) = 0$. Then

$$|\partial_\theta f(0)| \leqslant \int_0^{\theta_1} |\tau(f)(\theta')|\,\mathrm{d}\theta' \leqslant$$

$$\leqslant \int_0^{2\pi} |\tau(f)(\theta)|\,\mathrm{d}\theta \leqslant (2\pi)^{\frac{1}{2}}\left(\int_0^{2\pi} |\tau(f)(\theta)|^2\mathrm{d}\theta\right)^{\frac{1}{2}}$$

and therefore

$$|\partial_\theta f(0)|^2 \leqslant 2\pi\left(\int_0^{2\pi} |\tau(f)(\theta)|^2\mathrm{d}\theta\right)$$

By assumption the energy density attains its maximum at $\theta = 0$ so

$$E(f) \leqslant \pi\,|\partial_\theta f(0)|^2 \leqslant 2\pi^2 \int_0^{2\pi} |\tau(f)(\theta)|^2\mathrm{d}\theta.$$

To conclude this section two examples of how one can combine the results of this and the preceding section.

Define $i : M \to \mathbb{R}_+$ by

$i(p) = \sup\{r : \exp_p : T_pM \to M$ restricted to $B_r(0) \subset T_pM$ is a diffeomorphism onto $B_r(p)\}$.

Case 1

There exists a compact $K \subset M$ such that $\kappa_{M \setminus K} \leqslant 0$ and $i\,|\,M \setminus K \geqslant 2r_0 > 0$. Then any solution of the heat equation with initial curve f such that $m\left(f, \dfrac{1}{2\pi^2}\right) < r_0$ has bounded image.

Proof. For any p such that $B_{3r_0}(p)$ lies wholly in $M \setminus K$, $B_{r_0}(p)$ has the property $P\left(\dfrac{1}{2\pi^2}\right)$. For let g be a closed C^2 curve with image in $B_{r_0}(p)$, w.l.o.g. assume that the maximum of $e(g)$ is attained at $\theta = 0$. $B_{2r_0}(g(0))$ is contained in $M \setminus K$ and so satisfies the hypotheses of Theorem 5A. Therefore Theorem 4A applies and the result follows.

Similarly:

Case 2

There exists a compact $K \subset M$ such that $\kappa_{M \setminus K} > 0$ and $i|_{M \setminus K} \geqslant 2r_0 \leqslant \leqslant (2\kappa_{M \setminus K}^{\frac{1}{2}})^{-1}$. Then any solution of the heat equation with initial curve f such that $m\left(f, \frac{1}{2\pi^2}\right) < r_0$ has bounded image.

6. EXISTENCE OF SOLUTIONS OF THE HEAT EQUATION

This section will be devoted to the following existence problem. Given a closed C^1 curve on M does there exist a solution f_t of the heat equation defined for $t \in [0, t_1)$, $t_1 \leqslant \infty$, such that f_t and $\partial_\theta f_t$ are continuous at $t = 0$ and f_t coincides with the given curve at $t = 0$?

This problem will be treated under the assumption that the manifold M satisfies a condition (for example one from the previous sections) which will ensure that any solution of the kind sought will have its image contained in a fixed compact set. A solution of the heat equation will henceforth mean a solution of this problem for a fixed but arbitrary f_0.

The main ideas of this treatment are those of [1]; therefore only an outline will be given with what is different here pointed out.

I. In [1] it is shown how to replace the harmonic map equation $\tau(f) = 0$ and the heat equation $\partial_t f_t = \tau(f_t)$ (which in terms of local coordinates on M are local systems of equations) with global systems. This is done as follows: M can be smoothly and properly embedded in some Euclidean space $\mathbb{R}^q$ by a map $w : M \to \mathbb{R}^q$. Given such an embedding it is always possible to construct a smooth Riemannian metric on a tubular neighbourhood N of M so that N is Riemannian fibred. Let $\pi : N \to M$ be the projection map and π^c_{ab} its covariant differential. Then

(a) A map $f : S^1 \to M$ satisfies $\tau(f) = 0$ if and only if the composition $W = w \circ f$ satisfies

$$\partial_\theta{}^2 W^c = \pi^c_{ab}\, \partial_\theta W^a \partial_\theta W^b . \tag{6.1}$$

(b) A deformation $f_t : S^1 \to M$ $(t_0 < t < t_1)$ satisfies $\tau(f_t) = \partial_t f_t$ if and only if $W_t = w \circ f_t$ satisfies

$$\partial^2_\theta W^c_t - \partial_t W^c_t = \pi^c_{ab}\, \partial_\theta W^a_t \partial_\theta W^b_t . \tag{6.2}$$

Also, given a smooth $W_t : S^1 \to N$ satisfying (6.2) for $t_0 \leqslant t < t_1$, if W_{t_0} maps S^1 into M then so does every W_t for $t_0 \leqslant t < t_1$.

II. *Derivative bounds*

LEMMA. *Any solution* $f_t : S^1 \to M$ *of the heat equation has energy density satisfying*

$$\partial_\theta^2 e(f_t) - \partial_t e(f_t) = |\tau(f_t)|^2. \tag{6.3}$$

Proof.

$$\partial_\theta^2 e(f_t) = \partial_\theta^2 \left(\frac{1}{2} \langle \partial_\theta f_t, \partial_\theta f_t \rangle\right) =$$

$$= \partial_\theta \langle \nabla_{\partial_\theta} \partial_\theta f_t, \partial_\theta f_t \rangle = \langle \nabla_{\partial_\theta} \nabla_{\partial_\theta} \partial_\theta f_t, \partial_\theta f_t \rangle +$$

$$+ \langle \nabla_{\partial_\theta} \partial_\theta f_t, \nabla_{\partial_\theta} \partial_\theta f_t \rangle.$$

The heat equation can be written $\partial_t f_t = \nabla_{\partial_\theta} \partial_\theta f_t$ so

$$\partial_t e(f_t) = \langle \nabla_{\partial_t} \partial_\theta f_t, \partial_\theta f_t \rangle = \langle \nabla_{\partial_\theta} \partial_t f_t, \partial_\theta f_t \rangle =$$

$$= \langle \nabla_{\partial_\theta} \nabla_{\partial_\theta} \partial_\theta f_t, \partial_\theta f_t \rangle .$$

Therefore

$$\partial_\theta^2 e(f_t) - \partial_t e(f_t) = \langle \nabla_{\partial_\theta} \partial_\theta f_t, \nabla_{\partial_\theta} \partial_\theta f_t \rangle = |\tau(f_t)|^2 .$$

Put $H(\theta_1, \theta_2, t) = \frac{1}{2} (\pi t)^{-\frac{1}{2}} \exp\left[\frac{-(\theta_1 - \theta_2)^2}{4t}\right]$ for $\theta_1, \theta_2 \in \mathbb{R}$, $t \in \mathbb{R}_+$. H is a fundamental solution for the operator $L_\theta = \partial_\theta^2 - \partial_t$, satisfies $L_{\theta_1} H = L_{\theta_2} H = 0$, and the identity

$$u(\theta_1, t) = -\int_{t_0}^{t} \mathrm{d}\tau \int_{\mathbb{R}} H(\theta_1, \theta_2, t-\tau)\, L_{\theta_2} u(\theta_2, \tau)\, \mathrm{d}\theta_2 + \int_{\mathbb{R}} H(\theta_1, \theta_2, t-t_0)\, u(\theta_2, t_0)\, \mathrm{d}\theta_2 \quad t_0 < t < t_1 \tag{6.4}$$

holds for all u_t defined on S^1 which are of class C^2 in θ and C^1 in t for $t_0 \leqslant t \leqslant t_1$.

Suppose f_t is a solution of the heat equation defined for $0 \leqslant t < t_1$. Since $H > 0$ there follows from (6.3) and (6.4)

(6.5) $$e(f_t)(\theta_1) \leqslant \int_{\mathbb{R}} H(\theta_1, \theta_2, t - t_0) e(f_{t_0})(\theta_2) \mathrm{d}\theta_2 \text{ for } 0 < t_0 < t < t_1.$$

For $t > 1$, putting $t - 1$ for t_0 in (6.5) there follows

$$e(f_t)(\theta_1) \leqslant \int_{\mathbb{R}} H(\theta_1, \theta_2, 1) e(f_{t-1})(\theta_2) \mathrm{d}\theta_2$$

and therefore

$$e(f_t) \leqslant \text{const} \int_0^{2\pi} e(f_{t-1})(\theta) \mathrm{d}\theta.$$

Any smaller value can be put in for $t - 1$ on the right for example zero since $e(f_t)$ is assumed to be continuous at $t = 0$.

For $0 < t \leqslant 1$ put $t_0 = 0$ in (6.5) to obtain

(6.6) $$e(f_t)(\theta_1) \leqslant \int_{\mathbb{R}} H(\theta_1, \theta_2, t) e(f_0)(\theta_2) \, \mathrm{d}\theta_2.$$

Put $\overline{e}(f_0) = \sup_{\theta \in S^1} e(f_0)(\theta)$ then (6.6) shows that

$$e(f_t)(\theta) \leqslant \text{const.}\, \overline{e}(f_0).$$

To summarize:

THEOREM 6A. *Let f_t be a solution of the heat equation for $0 \leqslant t < t_1$. Then*

$$e(f_t) \leqslant \text{const.} \int_0^{2\pi} e(f_0)(\theta) \, \mathrm{d}\theta \quad \text{for } 1 \leqslant t < t_1$$

$$e(f_t) \leqslant \text{const.} \sup_{\theta \in S^1} e(f_0)(\theta) \qquad \text{for } 0 \leqslant t \leqslant 1$$

with the constants not depending on f_t.

The difference between this and the derivation of bounds for the first order space derivatives in [1] is that the curvature terms in the identity (6) §8A in [1] do not appear in the corresponding identity here (i.e. (6.3)), because the domain is 1-dimensional. It is therefore not necessary to impose curvature restrictions on the target manifold M.

For a solution f_t of the heat equation there is the following theorem, proved in [1], for the second derivative with respect to θ of the W_t in (6.2).

THEOREM 6B. *Given ϵ there is a constant C independent of t such that $|\partial_\theta^2 W_t^c| < C$ for $t > \epsilon$. The constant C depends on f_0.*

This is proved by using the formula

$$W_t^c(\theta) = -\int_0^t \mathrm{d}\tau \int_{\mathbb{R}} H(\theta, \theta', t-\tau)\, F^c(\theta', \tau)\, \mathrm{d}\theta' + W_0^c(\theta, t)$$

where

$$W_0^c(\theta, t) = \int_{\mathbb{R}} H(\theta, \theta', t)\, W^c(\theta', 0)\, \mathrm{d}\theta'$$

and the F^c the functions on the right of (6.2) and the properties of the fundamental solution H. The only remark that should be made is that here it is assumed that the manifold satisfies conditions which will ensure that the image of any solution will be contained in a fixed compact set and therefore the embedding conditions in [1] are not necessary since the inequalities (12), §8D in [1] are automatically satisfied on a compact set.

III. The following two theorems are proved in §10 [1].

THEOREM 6C. *If f_t and f_t' are two solutions of the heat equation with $f_0 = f_0'$ then they coincide for all relevant $t > 0$.*

THEOREM 6D. *Let M' be a compact subset of M. Then for any closed C^1 curve $f_0 : S^1 \to M$ such that $f_0(S^1)$ lies in M' there is a positive constant t_1 depending only on M' and the magnitude of the energy density $e(f_0)$ such that there exists a solution f_t for that f_0 for $0 \leqslant t \leqslant t_1$.*

From these two theorems one deduces the following which corresponds to theorem 10C in [1].

THEOREM 6E. *There is a unique solution f_t of the heat equation defined for all $t \geqslant 0$.*

Proof. Such a solution exists for small t by Theorem 6D and is unique by Theorem 6C. Let t_1 be the largest number such that a solution of the kind sought exists for $0 \leqslant t < t_1$ and suppose that t_1 is finite. By assumption the manifold M satisfies conditions which ensure that the images $f_t(S^1)$ $(0 \leqslant t < t_1)$ all lie in a compact subset of M. Theorem 6A shows that the energy density remains bounded and therefore by Theorems 6C and 6D there is a fixed positive number ϵ such that any f_t can be continued as a solution into the interval $(t, t+\epsilon)$. This contradicts the finiteness of t_1.

7. SUBCONVERGENCE OF SOLUTIONS

In this section let f_0 be a fixed closed C^1 curve $S^1 \to M$ and assume that M satisfies conditions which will ensure that any solution of the heat equation f_t which is continuous along with $\partial_\theta f_t$ at $t = 0$ and which coincides with the given f_0 at $t = 0$ will have its image contained in a fixed compact set. Then by the preceding section a unique solution f_t exists for all $t \in [0, \infty)$. In this section a proof of the following:

THEOREM 7A. *There is a sequence $t_1, t_2, t_3, \ldots$ with $t_k \to \infty$ such that the curves $f_k = f_{t_k}$ converge uniformly to a closed geodesic f.*

LEMMA. (See [3]). *For* $k(f_t) = \dfrac{1}{2}\langle \partial_t f_t, \partial_t f_t \rangle$ *one has the following identity*

$$\partial_t k(f_t) = \partial_\theta^2 k(f_t) - |\nabla_{\partial_\theta} \partial_t f|^2 + \langle R(\partial_\theta f_t, \partial_t f_t)\partial_\theta f_t, \partial_t f_t \rangle. \tag{7.1}$$

Proof.

$$\partial_\theta^2 \left[\frac{1}{2}\langle \partial_t f_t, \partial_t f_t \rangle\right] = \partial_\theta \langle \nabla_{\partial_\theta} \partial_t f_t, \partial_t f_t \rangle =$$

$$= \langle \nabla_{\partial_\theta} \nabla_{\partial_\theta} \partial_t f_t, \partial_t f_t \rangle + \langle \nabla_{\partial_\theta} \partial_t f_t, \nabla_{\partial_\theta} \partial_t f_t \rangle =$$

$$= \langle \nabla_{\partial_\theta} \nabla_{\partial_t} \partial_\theta f_t, \partial_t f_t \rangle + |\nabla_{\partial_\theta} \partial_t f_t|^2.$$

Also,

$$\partial_t \left[\frac{1}{2}\langle \partial_t f_t, \partial_t f_t \rangle\right] = \langle \nabla_{\partial_t} \partial_t f_t, \partial_t f_t \rangle =$$

$$= \langle \nabla_{\partial_t} \nabla_{\partial_\theta} \partial_\theta f_t, \partial_t f_t \rangle.$$

Combining these it is easily seen that

$$\langle \nabla_{\partial_t} \nabla_{\partial_\theta} \partial_\theta f_t, \partial_t f_t \rangle = \partial_\theta^2 \left[\frac{1}{2} \langle \partial_t f_t, \partial_t f_t \rangle \right] -$$

$$- | \nabla_{\partial_\theta} \partial_t f_t |^2 + \langle \nabla_{\partial_t} \nabla_{\partial_\theta} \partial_\theta f_t - \nabla_{\partial_\theta} \nabla_{\partial_t} \partial_\theta f_t, \partial_t f_t \rangle$$

which is (7.1).

LEMMA.

$\partial_t E(t) \to 0$ *as* $t \to \infty$. (*As before* $E(t) = E(f_t)$).

Proof. Put $K(f_t) = \int_0^{2\pi} k(f_t)(\theta)\, d\theta$ with $k(f_t)$ as in preceding lemma. $\partial_t E(t) = - 2K(f_t)$ (Corollary of Lemma 3A) and

$$\partial_t^2 E(t) = -2 \int_0^{2\pi} \partial_t k(f_t)(\theta)\, d\theta .$$

The last term in (7.1) is uniformly bounded for t greater than any given positive number by Theorems 6A and 6B. By integrating (7.1) over S^1 it therefore follows that there exists a constant C such that $\partial_t^2 E(t) \geqslant C$. $\partial_t E(t)$ cannot be bounded away from zero because it is integrable so if $C \geqslant 0$ then obviously $\partial_t E(t) \to 0$. Suppose $C < 0$. If for some $\epsilon > 0$ there exist arbitrarily large t_0 such that $\partial_t E(t_0) = -\epsilon$ then

$$\int_{t_0 + \epsilon/C}^{t_0} \partial_t E(t) \mathrm{d}t \leqslant \int_{t_0 + \epsilon/C}^{t_0} (-\epsilon - C(t_0 - t)) = \epsilon^2 / 2C$$

but this contradicts the integrability of $\partial_t E(t)$.

Remark. This lemma will be used in the proof of Theorem 7A. In [1] the corresponding result (Corollary §6(C)) is proved with the assumption that the target manifold has non-positive sectional curvature. This assumption is not necessary here.

In the following proof it will be convenient to use the function G defined by

$$G(\theta, \theta') = \int_0^{2\pi} \chi_{(\theta, 2\pi)}(r) \chi_{(0,r)}(\theta')\, \mathrm{d}r$$

where $\chi_{(a,b)}$ is the characteristic function of the interval (a, b). The formula

$$h(\theta) = h(0) - \partial_\theta h(0)(2\pi - \theta) - \int_0^{2\pi} G(\theta, \theta')\partial_\theta^2 h(\theta')\mathrm{d}\theta'$$

holds for all C^2 functions $h : S^1 \to \mathbb{R}$ and

$$\partial_\theta^2 \int_0^{2\pi} G(\theta, \theta')h(\theta')\mathrm{d}\theta' = -h(\theta) \tag{7.2}$$

holds for all continuous functions $h : S^1 \to \mathbb{R}$.

Proof of Theorem 7A. Let $W_t = w \circ f_t$ be the solution of (6.2) which corresponds to f_t. The mappings W_t and $\partial_\theta W_t$ form bounded equicontinuous families (Theorems 6A and 6B). Therefore there exists a sequence $t_1, t_2, t_3, \ldots$ with $t_k \to \infty$ such that the mappings $W_k = W_{t_k}$ converge uniformly along with $\partial_\theta W_k$ to a continuously differentiable mapping W. The W_k can be represented by the formula

$$W_k^c(\theta) = W_k^c(0) - \partial_\theta W_k^c(0)(2\pi - \theta) - \int_0^{2\pi} G(\theta, \theta')\partial_\theta^2 W_k^c(\theta')\mathrm{d}\theta'$$

or

$$\begin{aligned} W_k^c(\theta) &= W_k^c(0) - \partial_\theta W_k^c(0)(2\pi - \theta) - \\ &\quad - \int_0^{2\pi} G(\theta, \theta')(F_k^c(\theta') + \partial_t W_k^c(\theta'))\,\mathrm{d}\theta' \end{aligned} \tag{7.3}$$

where $F_k^c = \pi_{ab}^c\, \partial_\theta W_k^a \partial_\theta W_k^b$. By the preceding lemma the $\partial_t W_k^c(\theta)$ converge in the mean to zero as $k \to \infty$. Therefore, since G is bounded

$$\lim_{k\to\infty} \int_0^{2\pi} G(\theta, \theta')\, \partial_t W_k^c(\theta')\,\mathrm{d}\theta' = 0.$$

Passing to the limit in (7.3) there results for the mapping W

$$W^c(\theta) = W^c(0) - \partial_\theta W^c(0)(2\pi - \theta) - \int_0^{2\pi} G(\theta, \theta')F^c(\theta')\,\mathrm{d}\theta'$$

where $F^c(\theta') = \lim_{k\to\infty} F_k^c(\theta') = \pi_{ab}^c(W)\partial_\theta W^a \partial_\theta W^b$. Referring to (7.2) it is seen that W satisfies (6.1) which means that it corresponds to a closed geodesic.

REFERENCES

[1] J. EELLS and J.H. SAMPSON, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math., 86 (1964), 109 - 160.
[2] D. GROMOLL, W. KLINGENBERG and W. MEYER, *Riemannsche Geometrie im Grossen*, Lecture Notes in Mathematics 55, Springer 1968.
[3] R.S. HAMILTON, *Harmonic maps of manifolds with boundary*, Lecture Notes in Mathematics 471, Springer 1975.

Manuscript received: December 2, 1984.